Journal of Statistical Physics, Vol. 14, No. 5, 1976

Moments of the Montroll–Weiss Continuous-Time Random Walk for Arbitrary Starting Time

J. K. E. Tunaley[1]

Received July 1, 1975

The generalized version of the Montroll–Weiss formalism for continuous-time random walks is employed to show that some of the asymptotic results for large times appropriate to the ordinary walk become exact when the start of the observations is arbitrary.

KEY WORDS: Random walk; first waiting time distribution; asymptotic distribution.

The Montroll–Weiss equation[1] in its original form is appropriate to a continuous-time random walk in which it is certain that at time $t = 0$ the particle just hops into its "initial location." An attempt to generalize it to a situation where it is simply at its initial location at $t = 0$ has been made by Tunaley[2,3] by including the effect of a first waiting time distribution $H(t)$. However, the result presented is incorrect[4] and, using the notation in Ref. 2 together with the fact that

$$P\{N = 0\} = 1 - H(t) \tag{1}$$

[1] Centre for Radio Science and Physics Department, University of Western Ontario, London, Canada.

Eq. (4) in Ref. 2 should read for the generalized version

$$p(\mathbf{k}, s) = \frac{1 - h(s) + \lambda(\mathbf{k})(h(s) - \psi(s))}{s(1 - \lambda(\mathbf{k})\psi(s))} \tag{2}$$

[This now makes $p(\mathbf{k}, s)$ consistent with the generalized walk over random sites of Tunaley[5].] Clearly when $h(t) = \psi(t)$ the simple M–W form is recovered.

Asymptotic forms for the moments have been derived by Shlesinger[6] and for the actual distribution by Tunaley.[2] The scaling procedure in the latter is unaffected by the above change and since $\lim_{a\to\infty} \lambda(\mathbf{k}/a) = 1$, the asymptotic distributions which have been derived are also unchanged.

For arbitrary starting times it is necessary to express $h(s)$ in terms of $\psi(s)$. To accomplish this, we can note that in the M–W formalism the waiting times are independent of the individual jump vectors so that the hops form a sequence of events which is a simple renewal process. For such a process Feller[7] shows that if the process has been evolving for an infinite time beforehand,

$$h(s) = [1 - \psi(s)]/\alpha s \tag{3}$$

where α is the mean sojourn time, which is supposed to be finite. It has been argued[8] that this choice of $h(s)$ does not include the effects of the site location and that what is required is $h(s, \mathbf{r}_i)$ say, where $\mathbf{r}_i$ is the position vector of the starting location. Thus an examination of the following joint probability is required:

$$\begin{aligned} &P\{\text{particle is located at site with vector } \mathbf{r}_i \text{ at } t = 0 \text{ and} \\ &\quad \text{makes first jump in interval } (t, t + dt)\} \\ &\quad = P\{\text{makes first jump in } dt | \text{at } \mathbf{r}_i \text{ at } t = 0\} \\ &\quad\quad \times P\{\text{at } \mathbf{r}_i \text{ at } t = 0\} \end{aligned} \tag{4}$$

However, in physical terms all sites have the same properties so that the first term on the rhs should not be a function of $\mathbf{r}_i$.[5] Of course this may not be true in reality if the M–W walk is used as an approximation to more complex walks such as that employed by Scher and Lax[9] in the theory of ac conductivity.

The moments may be calculated using Shlesinger's method. It is easily shown that the Laplace transform with respect to time of the mean displacement is given by

$$\langle x(s) \rangle = \mu h(s)/s(1 - \psi(s)) \tag{5}$$

[here $h(s)$ replaces Shlesinger's $\psi(s)$ in the numerator] and the variance by

$$\mathrm{Var}(x(t)) = \mathscr{L}^{-1}\left\{\frac{\sigma^2 h}{s(1 - \psi)} + \mu^2\left(\frac{2\psi h}{s(1 - \psi)^2}\right)\right\} - \mu^2\left\{\mathscr{L}^{-1}\frac{h}{s(1 - \psi)}\right\}^2 \tag{6}$$

Using Eq. (3) with finite α, we find

$$\langle x(t) \rangle = \mu t/\alpha \tag{7}$$

and for symmetric walks ($\mu = 0$)

$$\langle x^2(t) \rangle = \mathrm{Var}(x(t)) = \sigma^2 t/\alpha \tag{8}$$

Equations (7) and (8) are in fact the asymptotic results for the ordinary M–W walk: With arbitrary starting time they hold exactly for all $t \geqslant 0$.

On the other hand, when the walk is asymmetric so that $\mu \neq 0$ the asymptotic results do not appear to become exact, except in the trivial case where $\psi(t)$ is an exponential density.

When the mean sojourn time is infinite it becomes difficult to express $h(s)$ in closed form. The discussion in Ref. 2 is still appropriate.

REFERENCES

1. E. W. Montroll and G. H. Weiss, *J. Math. Phys.* **6**:167 (1965).
2. J. K. E. Tunaley, *J. Stat. Phys.* **11**:397 (1974).
3. J. K. E. Tunaley, *J. Stat. Phys.* **12**:1 (1975).
4. S. Scott, private communication (1975).
5. J. K. E. Tunaley, submitted to *J. Stat. Phys.*
6. M. F. Shlesinger, *J. Stat. Phys.* **10**:421 (1974).
7. W. Feller, *An Introduction to Probability Theory and its Applications*, Vol. 2, 2nd ed., Wiley, New York (1971).
8. H. Scher, private communication (1975).
9. H. Scher and M. Lax, *Phys. Rev. B* **7**:4491 (1973).